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Critical point of infinite type in one dimension

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Abstract. An exact solution displaying a critical point of infinite type is given in one dimension. The properties are similar to those of a regular critical point, but with some quantitative differences.

1. Introduction

Recently the concept of critical point of infinite type has been introduced and its relation with exact models has been shown (Benguigui 1977). Its properties were defined as the limit of the properties of a critical point of type t (Schulman 1973) when $t \to \infty$. We recall that a critical point of type t may be described by the Landau–Ginsburg–Wilson Hamiltonian

$$\mathcal{H}_t = \int [aP^2(\mathbf{r}) + bP^{2t}(\mathbf{r}) + c(\nabla P)^2] \, \mathrm{d}_d V \tag{1}$$

where P is the normalised order parameter. As habitual $a = A(T - T_0)$, and b and c are temperature independent. T_0 is the bare transition temperature and is in general different from the true transition temperature. However in the mean field approximation T_0 becomes the transition temperature.

In dimension $d \geqslant 2$, the critical point of infinite type has remarkable properties; (1) the mean field solution is exact even for $d = 2$ and consequently T_0 is, in this case, the transition temperature, (2) below T_0 the order parameter $\langle P \rangle$ is constant and equal to 1, i.e. the ordering is perfect, (3) the susceptibility and the specific heat are null below T_0, but both diverge in the disordered phase ($T \to T_{0+}$).

Another way to define a critical point of infinite type is to put $t = \infty$, in the Hamiltonian (1). Thus, in this case, the Hamiltonian reduces to

$$\mathcal{H}_\infty = \int [aP^2(\mathbf{r}) + c(\nabla P)^2] \, \mathrm{d}_d V \tag{2}$$

but with $P^2 \leqslant 1$. It should be interesting to study the properties of this Hamiltonian in three dimensions, but it does not have an exact solution. As shown by several authors (Lajzerowicz and Pfeuty 1971, Scalapino *et al* 1972, Dieterich 1976) an exact solution can be found in one dimension, although there is no phase transition. The system becomes ordered only at $T = 0$. In this paper we present the properties derived from $\mathcal{H}_\infty$ in one dimension. The free energy F_0 is, in the thermodynamic limit, the lowest eigenvalue of the Schrödinger equation:

$$-\frac{1}{2m}\frac{\partial^2\psi}{\partial P^2}+aP^2\psi=F\psi \tag{3}$$

($m=2c/k^2T^2$). In our case we have to add the condition $\psi=0$ for $P^2\geqslant 1$, since the potential is infinite for $P^2\geqslant 1$. The quantum-mechanical problem described by (3) is that of a particle in a box, with the potential inside the box equal to aP^2 (figure 1). The eigenvalues of this Hamiltonian have been calculated by Rotbart (preceding paper) and we shall use his results. The free energy is expressed in dimensionless units $\epsilon_0=F_0(8c)/k^2T^2$, as a function of $K=8ac/k^2T^2$. In figures 2, 3 and 4, we show, respectively, ϵ_0, $\mathrm{d}\epsilon_0/\mathrm{d}K$ and $\mathrm{d}^2\epsilon_0/\mathrm{d}K_2$, as functions of K.

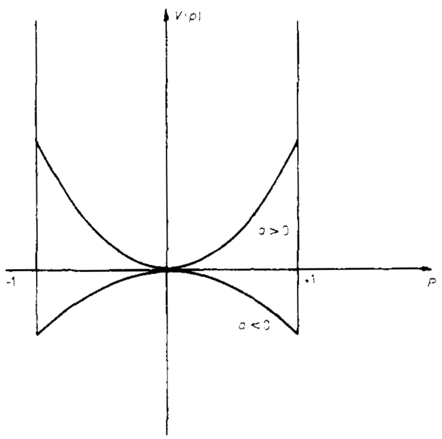

Figure 1. Potential of equation (3) as a function of P for $a>0$ and $a<0$.

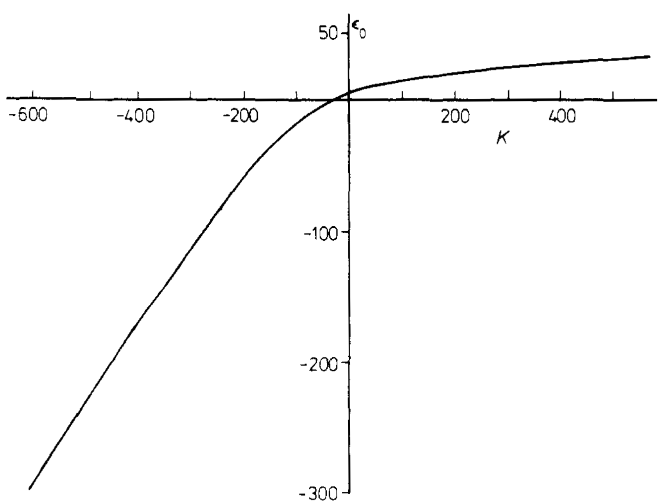

Figure 2. The dimensionless free energy ϵ_0 as a function of $K=8ac(kT)^{-2}$.

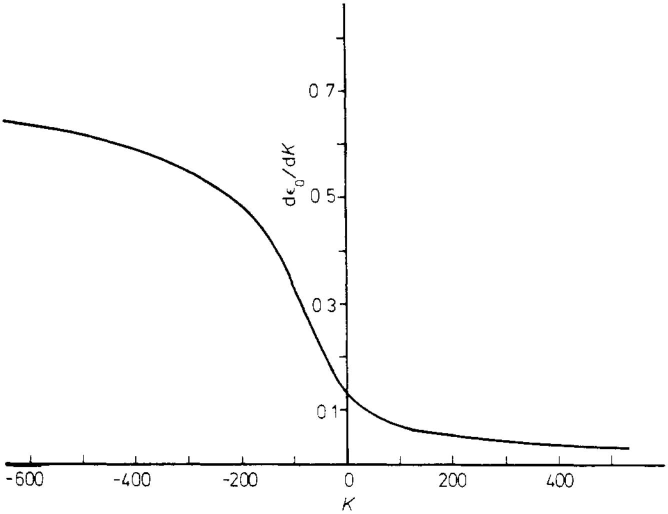

Figure 3. First derivative $d\epsilon_0/dK$ as a function of K. For $K \to -\infty$ $(T \to 0)$ $d\epsilon_0/dK$ goes to 1.

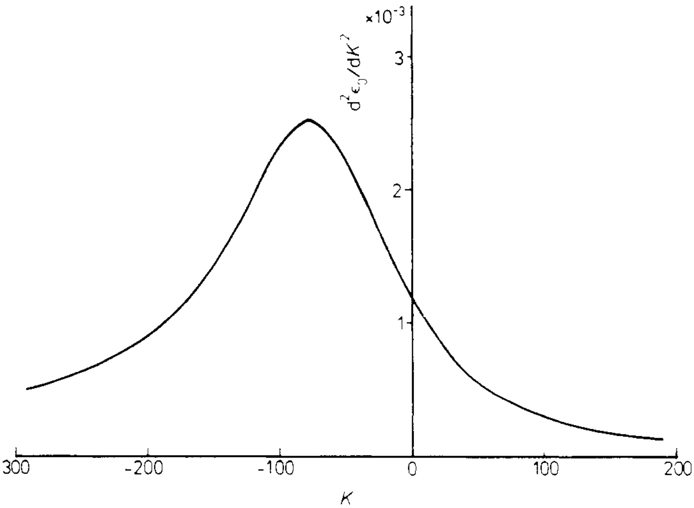

Figure 4. Second derivative $d^2\epsilon_0/dK^2$. If $T \sim T_0$, the curve gives the variation of the specific heat as a function of $T - T_0$.

2. Discussion

If we suppose that $T \sim T_0$ and that the variation of K is essentially given by the parameter $a = A(T - T_0)$, the curves give the properties of the system near T_0, and K is directly proportional to T. $d\epsilon_0/dK$ is equal to $(dF_0/da) = \langle P^2 \rangle$. In figure 3 one can see the beginning of the saturation of $\langle P^2 \rangle$ for $K \ll 0$, since as shown below $\langle P^2 \rangle \to 1$ for $T \to 0$. The second derivative $d^2\epsilon_0/dK^2$ is equal to $C/(8c/k^2T^2)$, where C is the specific heat. Contrary to the case $t = 2$ (Scalapino *et al* 1972) the specific heat exhibits a strong maximum below T_0 $(K < 0)$. However, as in the case $t = 2$, the one dimensional behaviour is reminiscent of mean field behaviour: for $t=\infty$, C diverges for $d \geqslant 2$, and for $d=1$ here we see that C has a marked maximum.

At low temperature ($K \ll 0$), an approximate solution of (3) can be found. First we can consider each minimum of the potential (figure 1) as independent, i.e. we neglect the tunnelling. Furthermore, the minima are deep enough that the ground state can be calculated by taking the potential near $P=\pm 1$ to be a straight line. The solution of this quantum-mechanical problem is given by Goldman *et al* (1964) and one finds that the ground state is

$$F_0 = [a + 2{\cdot}34(|a|k^2T^2/2c)^{1/3}] \tag{5}$$

or in terms of ϵ and K,

$$\epsilon = K + 3{\cdot}75|K|^{2/3}. \tag{6}$$

When K is strongly negative we have $\epsilon_0 \sim K$, $\langle P^2\rangle \sim 1$ and $F_0 \sim a$ as given by the classical mean field solution. As suggested by Dieterich (1976) one can define a critical region, outside of which the free energy is practically equal to the value given by mean field calculations. From (5) to (6), one must write $|K| \gg (3{\cdot}75)^3 \approx 50$. If again one assumes that $T \sim T_0$, the critical region ΔT is given by

$$\Delta K = \frac{8Ac\,\Delta T}{(kT_0)^2} \approx 50 \tag{7}$$

or

$$\Delta T \approx 4\frac{(kT_0)^2}{Ac}. \tag{8}$$

This expression may be compared with the critical region of the regular ($t=2$) critical point $\Delta T = (bkT_0)^{2/3}/Ac^3$. For $t=2$ the parameter c appears cubed in the expression for ΔT. The value of the critical region depends strongly on c, which is not the case for $t=\infty$.

The correlation length is given (Scalapino *et al* 1972) by $\xi = kT/(F_1 - F_0)$ when F_1 and F_0 are the first two eigenvalues of the Schrödinger equation. If T is low enough, we can use the WKB method to calculate the splitting of the levels (Landau and Lifshitz 1966). We find that, at low temperature ($K \ll 0$)

$$F_1 - F_0 \sim \frac{|a|^{1/2}}{\pi m^{1/2}} \exp\frac{\pi F_0 m^{1/2}}{|a|^{1/2}}. \tag{9}$$

Since $m^{1/2} = \sqrt{2c}/kT$, we have

$$\xi = \frac{\pi\sqrt{2c}}{|a|^{1/2}} \exp\frac{\pi\sqrt{2c}}{|a|^{1/2}kT}. \tag{10}$$

As is expected, ξ increases exponentially with decreasing temperature. The increase is more rapid than in the case $t=2$. Dieterich has shown that for $t=2$ we have

$$\xi = \left(\frac{c}{|a|}\right)^{1/2} \frac{T}{\bar{T}} \exp\frac{\bar{T}}{T} \tag{11}$$

where $\bar{T}$ is dependent on a, b and c. In (11) there is a factor proportional to T multiplying the exponential, which is not present in expression (10).

Qualitatively, we see that the properties of the critical point of infinite type in one dimension are similar to those of the regular critical point ($t = 2$). This is clearly because of the one-dimensional aspect of the problem, for which no ordered state is permitted at finite temperature. However, quantitatively these are some differences. The specific heat has a strong maximum; the critical region does not depend too strongly on c; the correlation length, when $T \to 0$, increases very rapidly. One important difference is that the Hamiltonian $\mathscr{H}_\infty$ can be used down to $T = 0$, where $\langle P^2 \rangle = 1$, as is expected. This behaviour is observable on the curve $d\epsilon_0/dK$ as a function of K, where one can see the beginning of the saturation (figure 3).

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